COMPARISON OF TEACHING METHODS IN EARLY ALGEBRA

Cyril Quinlan, fms, Australian Catholic University

This paper reports ongoing research on mental processes in early algebra. Two Year 7 classes were each taught with the aim of developing an understanding of algebraic generalizations which included the distributive law. One class used arithmetic examples leading to generalizations. The other used an objects-and-containers model to assist. Significantly better gains were recorded by the latter class on attitudes and content-specific achievement. No significant differences were detected before the teaching intervention for these mixed ability coeducational classes. The evidence points to the likelihood that the use of a concrete analogue assisted cognitive development.

This paper presents a report on one stage of an ongoing research project. The author invites comment from colleagues. The project is attempting to throw light on the elusive goal of understanding mental processes used by students when they first meet basic principles in the algebra of generalized arithmetic. One aspect of this search is to try to tease out the influence(s), whether for benefit or hindrance, of appropriate concrete analogues when they are used as instructive aids in early algebra. Central to the structure of the research stage described herewith was having the same teacher teach her two Year 7 classes by different methods. One class was lead towards the acceptance and understanding of algebraic generalizations by the study of sets of arithmetic examples (Method A), whereas the other was led in a similar direction with the aid of concrete manipulatives (Method B). The concrete model chosen was the objects-and-containers model as described in Quinlan, Low, Sawyer, and White (1993), Unit 1 Worksheet 3.

The researcher was well-aware that research focused on comparing teaching approaches or assessing the influence of intervention teaching does not always produce a significant result.

> Brophy and Good (1986, p. 329) said of projects prior to the 1970's that 'there has been remarkably little systematic research linking teacher behavior to student achievement.' In the 1970's, the well-resourced projects Developing Mathematical Processes and Individually Guided Education (Romberg, 1977) produced the outcome that 'little evidence is available to substantiate the importance of teacher actions', according to Romberg and Collis (1987, p. 17). These researchers identified the importance of including observations of teacher actions, pupil actions, and teacher-pupil interactions for productive research. Investigators are challenged not only by these aspects but also by the need to balance characteristics of schools, teachers, classroom groups, and individual students when comparing teaching approaches.
>
> (Quinlan, 1992, p. 16)

However, encouragement came from the fact that, despite the difficulties, successful research in the area has been documented. Brophy and Good (1986) summarized examples of progress made since 1970 in Process-Product Research as well as Correlational and Experimental Studies.

Sweller has been a co-author of several papers (cf. Owen & Sweller, 1985; Ward & Sweller, 1990) reporting a variety of recent research projects which identified significant effects of teaching strategies. Presmeg (1986; 1991) documents the interaction between visual learners and teaching styles which varied according to the degree of visualization employed.

Procedure

The two Year 7 classes of subjects in this project completed the same test as a Pretest and a Posttest in November 1993. The majority of the test items were used in Quinlan's 1992 doctoral research. The researcher prepared worksheets for each group and discussed them at length with the teacher prior to the teaching intervention. They were a revised and extended version of similar worksheets used by the researcher in an interview setting at another school in May 1993.

The November project may be categorized as an Experimental Study since the following procedures allowed the project to concentrate on the use of two different teaching approaches:

(a) The two classes involved in the study were both of mixed ability;

(b) Both were coeducational classes;

(c) The same teacher taught each class;

(d) The teacher was the usual mathematics teacher for each class;

(e) Each class had similar background experiences in algebra, having had a few lessons in algebra in April without any substantial follow-up;

(f) The intervention teaching was for four periods in each class;

(g) The Pretests were done on the same day by each class;

(h) The Posttests were done on the same day by each class, exactly a fortnight after the Pretests and at the same time of day as the Pretests;

(i) The worksheets were similar for the two classes in terms of some common exercises and the sequence of development. Differences centred on contrasts in teaching approaches;

(j) Each class worked in groups;

(k) The teacher kept to a problem-solving approach in each class, giving groups time to discuss ideas and exercises before directing them towards intended goals;

(l) In each class, students presented their ideas on the blackboard or at the overhead projector as the discussions progressed;

(m) Each worksheet included open-ended exercises such as

Describe (rewrite) the following in as many ways as you like;

(n) Each worksheet used a variety of tasks to lead the students to accept and understand that simple algebraic expressions may be written in several equivalent forms, for example,

$$2y + 8 = 2(y + 4) = (y + 3) + (y + 5);$$

(o) Each worksheet used a variety of tasks to lead the students to accept and understand that the letter symbols in this form of algebra always stand for numbers and that the numbers may vary.

Copies of the worksheets may be obtained from the author. A brief outline now follows.

The Method A class were led to establish four "conclusions" such as

$$2(3 + 4) = 2 \times 3 + 2 \times 4 \quad \text{and} \quad 2(6 + 4) = 2 \times 6 + 2 \times 4$$

by the use of arithmetic. They were then to describe or show the pattern(s) they saw. The hope was that they might recognize the aspects in common and come to regard the changing numbers in the pattern as variables of some sort. They were next invited to rewrite various expressions "in as many different ways as you like". These expressions were at first arithmetic, such as $2(5 + 4)$ and $(2 \times 7) + 6$, and later algebraic, such as $2(y + 5)$ and $5(y + 2)$. Space was provided for the students to make up some more examples "using a letter (such as one of your initials) to represent any number." The above sequence was repeated for a pattern which could be generalized as $3(2 + x) = 3 \times 2 + 3 \times x$. Two exercises were inserted using ❑ to represent "a space for any number." Definitions were then presented for the mathematical terms product, factorize, expand, and simplify and exercises on all but "product" were given.

The final question was composed of two similar parts, the first of which was:

> This question is about discoveries made by three groups in a mathematics class.
> In each part of the question, you are to do two things:
> (a) Write **one equation** in algebra which would be true for the discoveries made by all three groups
> (b) **Explain** why you claim that your equation is true.
> [i] Group A discovered: $2\,(5 + 3) = 2 \times 5 + 6$
> Group B discovered: $2\,(0 + 3) = 2 \times 0 + 6$
> Group C discovered: $2\,(6.25 + 3) = 2 \times 6.25 + 6$
> (a) My equation: . . . .
> (b) My explanation: . . . .

The Method B groups followed a similar sequence of exercises with the continued assistance of the objects-and-containers model. The activities opened with the provision of a letter symbol "to represent the number of small objects placed inside any one of the containers provided." Then each group was asked to do the following exercises twice, using different y values each time:

> Choose a value of y and place y objects in each of two containers.
> (a) **Build** each of the following expressions using objects and containers.
> Above the expression **draw** a diagram of what you built, and
> **write** down the number of objects in each case:

 $y =$	$2y =$	$y + 4 =$	2 lots of $(y + 4)$ $= 2(y + 4) =$	$2y + 8 =$

> (b) What do you notice about the last two results? Why is this?

The worksheet asked students to rewrite the expressions given in Worksheet A. However, the first three cases were presented using the model in diagram form, as follows:

Describe in as many ways as you like the number of small objects represented in the model by:

____

As in Worksheet A, the above sequence was repeated for a pattern which could be generalized as $3(2+x) = 6+3x$ [$= 3 \times 2 + 3 \times x$]. The same exercises on defined terms were given except that, at first, the students were asked to use the model to show that they were correct, e.g.:

Build the following using the objects and containers model and write an expression in algebra to describe the number of objects represented in the arrangement shown.

represents ....

Rearrange what you have built so that you can **simplify** the expression. Draw the way you rearranged the materials and write your simplified expression beside your drawing.

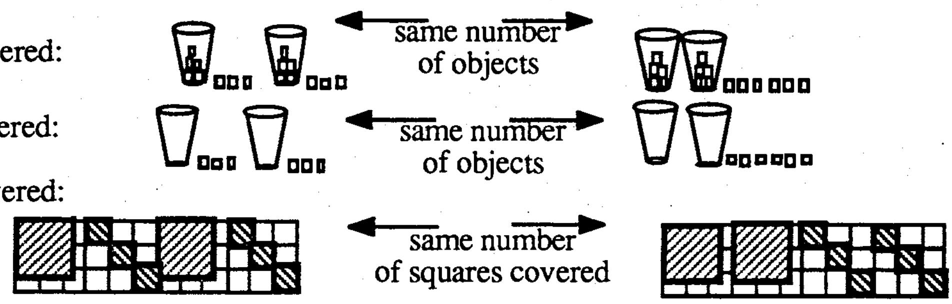
The final question paralleled that of the Method A worksheet except that it was presented in diagram form using both the objects-and-container and an area model, as shown:

[i]

Group A discovered:

Group B discovered:

Group C discovered:



Results and discussion

Pre- and post- test differences. According to t-test analyses, each class showed statistically significant progress in the following scale scores which were based on the Pre- and Post- test responses. The scales (c) and (d) are similar to those explained in Quinlan 1992 (pp. 162 -164):

(a) Total Test Score ($p = .001$ for Method A; .012 for Method B);

(b) Distributive Law Scale (alpha reliability coefficient = .63), as explained below. The Distributive Law was a major consideration in planning the worksheets ($p = .047$; .000);

(c) Generalized Number and/or Variable Scale ($\alpha = .87$), using 10 test items to measure the degree of understanding students have for the meaning of algebraic symbols ($p = .019$; .045);

(d) Acceptance of Lack of Closure Scale ($\alpha = .73$), measuring the degree of acceptance of symbols in 6 items without differentiating between correct and incorrect answers ($p = .000$; .000).

On favourable attitudes to algebra as measured by a 7-item Attitude Scale ($\alpha = .72$), the Method B class registered significant gains ($p = .012$) but Method A class did not.

Details of these *t*-tests will be available at the conference.

Differences between groups. Only one aspect of cognitive difference between groups has been identified. While there were no statistically significant differences between the two groups at the pretest stage, by the time of the posttest the Method B Group achieved significantly better than the Method A Group on a scale formed from the following five items related to the Distributive Law:

(i) If $y = 3$, what is the value of $2(y + 5)$?

(ii) Four lots of $n + 5$ is the same as: ...

(iii) Multiply $b + 5$ by 3.

(iv) Factorize $15 + 25t$.

(v) Expand and simplify $3(3 + x) + 2(2x + 5)$. Show your working.

The maximum scale score was 6 as the last item was scored out of 2, taking account of the two expansions and ignoring the step of simplifying. There were no significant differences on pretest scores for this scale but the posttest scores were significantly different ($t = 2.75$, $p = .008$). The effects of the type of instruction on the posttest scores were found to be significant ($F = 7.307$, $p = .010$) when an analysis of variance was completed using the pretest scores as covariates.

Attitudinal measures showed a significant difference in trends between the two classes. Prior to the algebra sessions no attitudinal differences emerged but after the experience of the four lessons, those who used the concrete approach registered significantly ($t = 2.52$, $p = .016$) more positive attitudes to algebra than did their counterparts. Using pretest measures as covariates showed significant effects of instruction method on posttest attitude scores ($F = 11.75$, $p = .002$).

The statistical details of *t*-tests and analyses of variance are given in Tables 1 and 2.

Table 1

Significant Differences Between Concrete & Arithmetic Groups on Posttest

Scale	Max	Mean Conc.	Mean Arith.	*t* value	*df*	*p*	Favours
Distributive Law	6	2.64 (1.40)	1.32 (0.96)	2.75 (1.12)	48 (48)	.008 (.269)	Conc. (-)
Attitude	28	19.30 (18.27)	16.47 (16.86)	2.52 (1.30)	42 (41)	.016 (.202)	Conc. (-)

Note. Matched, mixed ability groups. Max. = maximum possible score.
Conc. = Method B, Concrete Approach ($n = 25$ for Distributive Law and 23 for Attitude);
Arith. = Method A, Arithmetic Approach ($n = 25$ for Distributive Law and 21 for Attitude).
Numbers vary because of blanks on response sheets. Pretest results are shown in brackets.

In Table 2, the grand mean is the average of the posttest results for all students in the analysis. The statistics show that the methods of teaching had a significant effect on the outcomes: The F values for method effect are significant at $p = .010$ and $.002$ respectively, and the multiple R^2 values indicate that methods accounted for, respectively, 55.8% and 62.1% of the total variance (cf. Nie, Bent, Jenkins, Steinbrenner, & Hull, 1975, 404).

Table 2
Significant Analysis of Variance Outcomes for Posttest Scores Using Pretest Scores as Covariants

Scale	Max.	Grand mean	Deviations from grand mean				Effect of method			
			unadjusted		adjusted for pretest					Multiple
			Conc.	Arith.	Conc.	Arith.	F	df	p	R^2
Distributive Law	6	1.98	0.66	- 0.66	0.47	- 0.47	7.307	1,47	.010	.558
Attitude	28	18.05	1.75	-1.84	1.39	-1.46	11.75	1,36	.002	.621

Note. Matched, mixed ability groups. Max. = maximum possible score.
Conc. = Method B, Concrete Approach ($n = 25$ for Distributive Law and 20 for Attitude);
Arith. = Method A, Arithmetic Approach ($n = 25$ for Distributive Law and 19 for Attitude).
Numbers vary because of blanks on response sheets.

Discussion. As was expected, those following Method A had a real struggle to derive generalizations from arithmetic examples. Halford and Boulton-Lewis (1992) have proposed a hierarchy of mappings which could lead from arithmetic examples to the understanding of the algebraic generalization given by $a(b+c) = (a \times b) + (a \times c)$. They suggest that each of a series of correspondences should be learned so well that retrieval is automatic before progressing to the next . "The load imposed by one structure mapping must be reduced to zero before the next structure mapping is undertaken, otherwise the cumulative load will become excessive" (p. 204). Thus, in the case used in the worksheet, the correspondences such as $3 + 4 = 7$, $2(3 + 4) = 14$, and $2 \times 3 + 2 \times 4 = 14$ must be available by immediate recall before proceeding to the correspondence in the conclusion of each case, such as $2(3 + 4) = 2 \times 3 + 2 \times 4$. Such conclusions must be accepted before progressing to the desired general correspondence, namely that $2(y+4) = 2 \times y + 2 \times 4$. The students were urged to underline the numbers common to each conclusion and then to consider the numbers that changed from conclusion to conclusion. The teacher assisted them by beginning the second lesson with an overhead of the conclusions, suitably colour-coded.

The posttest outcome that Method B students performed better on distributive law items directs attention to the concrete analogue used for Method B. Quinlan and Collis (1990, pp. 445 - 448) discuss the suitability of the objects-and-containers model and show that, in the context of this project, it has the strengths of commutativity, transferability and isomorphism. It appears to have the qualities to match the principle enunciated by Boulton-Lewis and Halford (1991, p. 37):

> The value of a concrete representation is that it mirrors the structure of the concept and the child should be able to use the structure of the representation to construct a mental model of the concept.

The evidence points to the likelihood that this concrete analogue assisted cognitive development and was neither redundant (Sweller, 1993) nor a distraction (Halford, 1993). It is worth considering that, as depicted in Figure 1, the model contributes directly to the development of the desired conceptual understandings.

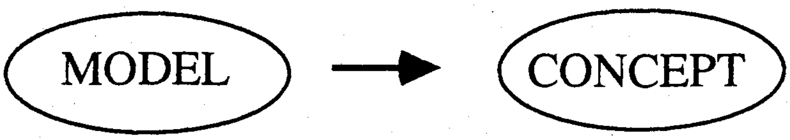


Figure 1. Model contributes directly to understanding of concept

Method B class spent more time on discussions (while working with concrete materials), whereas Method A class spent more time on writing answers. Leinhardt (1988, p. 141) drew attention to a potential role of concrete analogues which could have been relevant here:

> We need to explore more elegant ways of building consistent concrete representations that can serve as both an explanatory and exploratory system for children and to give them language tools for talking about such systems.

Sowell (1989) examined 60 research studies on the effects of manipulative materials in mathematics instruction during the 1960s and 1970s. Her comparison of the concrete versus abstract instructional condition for effects on achievement showed that "when treatments lasted a school year or longer, the result was significant in favor of the concrete instructional condition. Treatments of shorter duration did not produce statistically significant results" (p. 502). Furthermore, when instructional conditions were randomly assigned, "attitude measures were significant in favor of the concrete instructional condition" (p. 502). The November Quinlan project is an example of a short duration treatment which *did* have significant effects, both on performance and attitude. Attitudinal gains were no surprise: Generally favourable reactions from teachers and students to the use of manipulatives for algebra were reported in Quinlan et al. (1993) following four years of action research. As Collis and Biggs put it, "It seems that a well organised inter-modal strategy influences children's attitude to, as well as their comprehension of, the content being taught" (1991, p. 202).

A delayed posttest is planned for the students in the November 1993 study. Further data are to be collected from other participants and the possibility of interactions between outcomes and preferred thinking and perception modes is to be investigated.

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